Cosmological Adiabatic Geometric Phase of a Scalar Field in a Bianchi Spacetime*

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Abstract

A two-component formulation of the Klein-Gordon equation is used to investigate the cyclic and noncyclic adiabatic geometric phases due to spatially homogeneous (Bianchi) cosmological models. It is shown that no adiabatic geometric phases arise for Bianchi type I models. For general Bianchi type IX models the problem of the adiabatic geometric phase is shown to be equivalent to the one for nuclear quadrupole interactions of a spin. For these models nontrivial non-Abelian adiabatic geometrical phases may occur in general.

I Introduction

In Ref. [1] a two-component formulation of the Klein-Gordon equation is used to develop relativistic analogues of the quantum adiabatic approximation and the adiabatic dynamical and geometric phases. This method provides a precise definition of an adiabatic evolution of a Klein-Gordon field in a curved background spacetime. The purpose of this article is to employ the results of Ref. [1] in the investigation of geometric phases due to a spatially homogeneous background spacetime.

The phenomenon of geometric phase in gravitational systems has been previously considered by Cai and Papini [2], Brout and Venturi [3], Venturi [4, 5], Casadio and Venturi [6], Datta [7], and Corichi and Pierri [8]. The motivation for these studies extends from the investigation of the study of weak gravitational fields [2] to various problems in quantum cosmology and quantum gravity [3, 4, 5, 6, 7].

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These studies were mostly plagued by the problem of constructing appropriate inner products on the space of solutions of the Klein-Gordon equation. This problem has so far been solved for stationary spacetimes. Therefore the existing results have a limited domain of applicability. The most important feature of the method developed in Ref. [1] is that it avoids the above problem by showing that indeed the adiabatic geometric phase is independent of the choice of an inner product on the space of solutions of the field equation. This enables one to investigate the phenomenon of the adiabatic geometric phase for spatially homogeneous spacetimes which are clearly non-stationary. One must, however, note that the results presented in this article are relevant to adiabatically evolving Klein-Gordon fields [1]. This raises the question whether the adiabaticity of the evolution is compatible with the fact that the background spacetime is non-stationary. It turns out that the answer to this question is in the positive, i.e., there are Klein-Gordon fields in a non-stationary spacetime which do have adiabatic evolutions.

The organization of the paper is as follows. In section II, the results of Ref. [1] which will be used in this paper are briefly reviewed. The computation of the geometric phase for general spatially homogeneous (Bianchi) models are discussed in Section III. These are applied in the analysis of the geometric phase problem for Bianchi type I and type IX models in sections IV and V, respectively. A summary of the main results and the concluding remarks are given in section VI.

Throughout this paper the signature of the spacetime metric g is taken to be $(-,+,+,+)$. Letters from the beginning and the middle of the Greek alphabet are associated with an arbitrary local basis and a local coordinate basis of the tangent spaces of the spacetime manifold, respectively. The letters from the beginning and the middle of the Latin alphabet label the corresponding spatial components and take values in $\{1,2,3\}$.

II Two-Component Formalism and the Adiabatic Geometric Phase

Consider a complex scalar field Φ defined on a globally hyperbolic spacetime $(M,\mathrm{g})=(\mathbb{R}\times\Sigma,\mathrm{g})$ satisfying

$$\left(g^{\mu\nu}\nabla_\mu\nabla_\nu-\mu^2\right)\Phi=0\;, \tag{1}$$

where $g^{\mu\nu}$ are components of the inverse of the metric g, ∇_μ is the covariant derivative along $\partial/\partial x^\mu$ defined by the Levi Civita connection, and μ is the mass.

Denoting a time derivative by a dot, one can express Eq. (1) in the form

$$\ddot{\Phi}+\hat{D}_1\dot{\Phi}+\hat{D}_2\Phi=0\ , \tag{2}$$

where

$$\hat{D}_1 \ := \ \frac{1}{g^{00}}\left[2g^{0i}\partial_i-g^{\mu\nu}\Gamma^0_{\mu\nu}\right]\ , \tag{3}$$

$$\hat{D}_2 \ := \ \frac{1}{g^{00}}\left[g^{ij}\partial_i\partial_j-g^{\mu\nu}\Gamma^i_{\mu\nu}\partial_i-\mu^2\right]\ . \tag{4}$$

A two-component representation of the field equation (2) is

$$i\dot{\Psi}^{(q)}=\hat{H}^{(q)}\Psi^{(q)}\ , \tag{5}$$

where

$$\Psi^{(q)} \ := \ \left(\begin{array}{c} u^{(q)} \\ v^{(q)} \end{array}\right)\ , \tag{6}$$

$$u^{(q)} \ := \ \frac{1}{\sqrt{2}}\left(\Phi+q\dot{\Phi}\right)\ , \quad v^{(q)} \ := \ \frac{1}{\sqrt{2}}\left(\Phi-q\dot{\Phi}\right)\ , \tag{7}$$

$$\hat{H}^{(q)} \ := \ \frac{i}{2}\left(\begin{array}{cc} \frac{\dot{q}}{q}+\frac{1}{q}-\hat{D}_1-q\hat{D}_2 & -\frac{\dot{q}}{q}-\frac{1}{q}+\hat{D}_1-q\hat{D}_2 \\ & \\ -\frac{\dot{q}}{q}+\frac{1}{q}+\hat{D}_1+q\hat{D}_2 & \frac{\dot{q}}{q}-\frac{1}{q}-\hat{D}_1+q\hat{D}_2 \end{array}\right)\ , \tag{8}$$

and q is an arbitrary, possibly time-dependent, nonzero complex parameter.

Next consider the eigenvalue problem for $H^{(q)}$. Denoting the eigenvalues and eigenvectors by $E^{(q)}_n$ and $\Psi^{(q)}_n$, i.e.,

$$H^{(q)}\Psi^{(q)}_n=E^{(q)}_n\Psi^{(q)}_n\ , \tag{9}$$

one has [1]

$$\Psi^{(q)}_n=\frac{1}{\sqrt{2}}\left(\begin{array}{c} 1-iqE^{(q)}_n \\ 1+iqE^{(q)}_n \end{array}\right)\Phi^{(q)}_n\ , \tag{10}$$

where $\Phi^{(q)}_n$ satisfies

$$\left[\hat{D}_2-iE^{(q)}_n(\hat{D}_1-\frac{\dot{q}}{q})-\left(E^{(q)}_n\right)^2\right]\Phi^{(q)}_n=0\ . \tag{11}$$

This equation defines both the vectors $\Phi^{(q)}_n$ and the complex numbers $E^{(q)}_n$.

The following is a summary of some of the results obtained in Ref. [1].

1) Eq. (11) reduces to the ordinary eigenvalue equation for $\hat{D}_2$, if $\hat{D}_1=\dot{q}/q$. In this case $\Phi^{(q)}_n$ and $E^{(q)}_n$ are independent of the choice of q, and one can drop the labels (q) on the right hand side of Eq. (10).

2) $\Phi_n^{(q)}$ belong to the Hilbert space $\mathcal{H}_t = L^2(\Sigma_t)$ of square-integrable functions on the space-like hypersurfaces Σ_t where the integration is defined by the measure $[\det({}^{(3)}\mathrm{g})]^{1/2}$ and ${}^{(3)}\mathrm{g}$ is the Riemannian three-metric on Σ_t induced by the four-metric g.

3) Suppose that

   3.1) $\hat{D}_1 = \dot{q}/q$,

   3.2) $\hat{D}_2$ is self-adjoint with respect to the inner product $\langle \; , \; \rangle$ of $\mathcal{H}_t$,

   3.3) $\hat{D}_2$ has a discrete spectrum,

   3.4) during the evolution of the system $E_n \neq E_m$ iff $m \neq n$, i.e., there is no level-crossings, and

   3.5) E_n is a non-degenerate eigenvalue of $H^{(q)}$, alternatively E_n^2 is a non-degenerate eigenvalue of $\hat{D}_2$.

Then for all $m \neq n$, $\langle \Phi_m | \dot{\Phi}_n \rangle = \langle \Phi_m | \dot{\hat{D}}_2 \Phi_n \rangle / (E_n^2 - E_m^2)$. The relativistic adiabatic approximation corresponds to the case where the latter may be neglected. In this case, an initial two-component Klein-Gordon field

$$\Psi^{(q)}(0) = e^{i\alpha_n(0)} \Psi_n^{(q)}(0) + e^{i\alpha_{-n}(0)} \Psi_{-n}^{(q)}(0) \; , \tag{12}$$

with $n \geq 0$ and $\alpha_{\pm n}(0) \in \mathbb{C}$, evolves according to

$$\Psi^{(q)}(t) \approx e^{i\alpha_n(t)} \Psi_n^{(q)}(t) + e^{i\alpha_{-n}(t)} \Psi_{-n}^{(q)}(t) \; , \tag{13}$$

where $\alpha_{\pm n}(t) = [\alpha_n(0) + \alpha_{-n}(0)]/2 + \gamma_n(t) + \delta_{\pm n}(t)$, $\alpha_{\pm n}(0)$ are arbitrary constants, $\gamma_n(t)$ is the geometric part of both $\alpha_{\pm n}(t)$ and $\delta_{\pm n}(t)$ is the dynamical part of $\alpha_{\pm n}(t)$. They are given by

$$\gamma_n(t) = \int_{R(0)}^{R(t)} \mathcal{A}_n[R] \; , \tag{14}$$

$$\delta_{\pm}(t) = i\xi_n(t) \pm \frac{\eta_n(t)}{2} \; , \tag{15}$$

where

$$\mathcal{A}_n[R] := \frac{i\langle \Phi_n[R] | d | \Phi_n[R] \rangle}{\langle \Phi_n[R] | \Phi_n[R] \rangle} = \frac{i\langle \Phi_n[R] | \frac{\partial}{\partial R^a} | \Phi_n[R] \rangle}{\langle \Phi_n[R] | \Phi_n[R] \rangle} dR^a \; , \tag{16}$$

is the Berry's connection one-form [10], $R = (R^1, \cdots, R^n)$ are the parameters of the system, i.e., the components of the metric, d stands for the exterior derivative with

respect to R^a,

$$\xi_n(t) := \frac{1}{2}\int_0^t f_n(t')[1-\cos\eta_n(t')]dt' \;, \tag{17}$$

$$f_n(t) := \frac{d}{dt}\ln[q(t)E_n(t)] \;, \tag{18}$$

and η_n is the solution of

$$\dot\eta_n + f_n\sin\eta_n + 2E_n = 0\;, \qquad \eta_n(0) = \alpha_n(0) - \alpha_{-n}(0)\;. \tag{19}$$

In view of Eqs. (6), (7), and (10), Eq. (13) can be written in the form

$$\Phi = c\, e^{i\gamma_n(t)}(e^{i\delta_n(t)} + e^{i\delta_{-n}(t)})\Phi_n, \tag{20}$$

where Φ is the one-component Klein-Gordon field, i.e., the solution of the original Klein-Gordon equation (1), and $c := \exp[\alpha_n(0)+\alpha_{-n}(0)]$.

4) If in addition to the above adiabaticity condition one also has $\dot E_n \approx 0$, then an initial eigenvector $\Psi^{(q)}_n(0)$ evolves according to

$$\Psi^{(q)}(t) \approx e^{i\alpha_n(t)}\Psi^{(q)}_n(0)\;,$$

where the total phase angle α_n again consists of a geometric and a dynamical part. This case corresponds to what is termed as ultra-adiabatic evolution in Ref. [1].

Suppose that conditions 3.1) – 3.4) are satisfied, but E_n is $\mathcal{N}$ fold degenerate. Then the condition for the validity of the adiabatic approximation becomes $\langle\Phi^I_m|\dot\Phi^J_n\rangle \approx 0$, for all $m \neq n$ and $I, J = 1, 2, \cdots \mathcal{N}$. Here $\Phi^1_n, \cdots \Phi^{\mathcal{N}}_n$ are orthogonal eigenvectors spanning the degeneracy subspace of $\mathcal{H}_t$ associated with E_n^2. In this case, $\exp[i\alpha_{\pm n}(t)]$ become matrices of the form $e^{i\delta_{\pm n}(t)}\Gamma_n(t)$ where

$$\Gamma_n(t) := \mathcal{P}\,\exp[i\int_{R(0)}^{R(t)}\mathcal{A}_n]\;, \tag{21}$$

$\mathcal{P}$ is the path-ordering operator, $\mathcal{A}_n$ is a matrix of one-forms with components

$$\mathcal{A}^{IJ}_n[R] := \frac{i\langle\Phi^I_n[R]|d|\Phi^J_n[R]\rangle}{\langle\Phi^I_n[R]|\Phi^I_n[R]\rangle}\;. \tag{22}$$

In this case, a solution of the one-component Klein-Gordon equation (1) is given by

$$\Phi = \sum_{I,J=1}^{N}(c^I_n e^{i\delta_n(t)} + c^I_{-n}e^{i\delta_{-n}(t)})\Gamma^{IJ}_n(t)\Phi^J_n\;, \tag{23}$$

where $c^I_{\pm n}$ are constant coefficients determined by the initial conditions.

If the parameters R undergo a periodic change, i.e., $R(T) = R(0)$ for some $T \in \mathbb{R}^+$, then the path-ordered exponential $\Gamma_n(T)$ which is called the *cyclic adiabatic geometrical phase*, cannot be removed by a gauge transformation. Eq. (22) shows that the formula for the relativistic adiabatic geometric phase has the same form as its non-relativistic analogue [9]. In particular for $\mathcal{N} = 1$ (the non-degenerate case), one recovers Berry's connection one-form (16). In this case $\Gamma_n(T)$ reduces to an ordinary exponential and yields the Berry phase $e^{i\gamma(T)}$ for the Klein-Gordon field.

In the remainder of this paper, I shall consider the problem of the adiabatic geometric phase for a Klein-Gordon field minimally coupled to a spatially homogeneous gravitational field.

It is important to note that the cyclic geometric phase has physical significance, if one has a cyclic evolution[1]. For an adiabatic evolution, the evolving state undergoes a cyclic evolution, if the parameters of the system, in this case the components of the metric tensor, change periodically in time. This corresponds to the spatially homogeneous (Bianchi) cosmological models which admit periodic (oscillatory) solutions[2]. This observation does not however mean that the connection one-forms $\mathcal{A}_n$ and their path-ordered exponentials $\Gamma_n(t)$ lack physical significance for general nonperiodic Bianchi models. The cyclic adiabatic geometric phases have noncyclic analogues which occur in the evolution of any quantum state, [12, 13].

A noncyclic analogue of the non-Abelian cyclic geometric phase has recently been introduced by the present author [13]. In view of the results of Ref. [13], the *noncyclic adiabatic geometric phase* for an adiabatically evolving Klein-Gordon field is given by

$$\tilde{\Gamma}_n(t) := w_n(t)\Gamma_n(t) \,, \tag{24}$$

where $w_n(t)$ is an $\mathcal{N} \times \mathcal{N}$ matrix with entries:

$$w_n^{IJ}(t) := \langle \Phi_n^I[R(0)] | \Phi_n^J[R(t)] \rangle \,. \tag{25}$$

The noncyclic geometric phase has the same gauge transformation properties as the cyclic geometric phase. In particular, its eigenvalues and its trace are gauge-invariant physical quantities, [13].

If it happens that the connection one-form $\mathcal{A}_n$ is exact, i.e., it is a pure gauge, then there are two possibilities:

[1]For a discussion of the meaning of a cyclic evolution of a Klein-Gordon field see Ref. [1].
[2]For an example see Ref. [11].

a) The curve C traced by the parameters R of the system in time has a part which is a non-contractible loop. In this case the cyclic or noncyclic geometric phase will be a topological quantity analogous to the Aharonov-Bohm phase [10]. Such a geometric phase will be called a *topological phase*. Topological phases can occur only if the parameter space of the system has a nontrivial first homology group.

b) The curve C does not have a piece which is a non-contractible loop. In this case, the geometric phase is either unity (the cyclic case) or it depends only on the end points, $R(0)$ and $R(t)$, of C (the noncyclic case). Such a geometric phase will be called a *trivial geometric phase*.

III Spatially Homogeneous Cosmological Models

Consider Klein-Gordon fields in a spatially homogeneous (Bianchi) cosmological background associated with a Lie group G, i.e., $M = \mathbb{R} \times G$. In a synchronous invariant basis the spacetime metric g is given by its spatial components g_{ab}:

$$ds^2 = g_{\alpha\beta}\omega^\alpha\omega^\beta = -dt^2 + g_{ab}\omega^a\omega^b \ , \tag{26}$$

where ω^a are the left invariant one-forms and $g_{ab} = g_{ab}(t)$. Throughout this article I use the conventions of Ref. [14].

The first step in the study of the phenomenon of the adiabatic geometric phase due to a spatially homogeneous cosmological background is to compute the operators $\hat{D}_1$ and $\hat{D}_2$ of Eqs. (3) and (4) in the invariant basis. It is not difficult to see that with some care these equations are valid in any basis. One must only replace the coordinate labels $(\mu, \nu, \cdots, i, j, \cdots)$ with the basis (in this case invariant basis) labels $(\alpha, \beta, \cdots, a, b, \cdots)$, and interpret ∂_a as the action of the operators $\hat{X}_a$ associated with the dual vector fields to ω^a. This leads to

$$\begin{aligned} \hat{D}_1 &= g^{ab}\Gamma^0_{ab} \ , &(27)\\ \hat{D}_2 &= -\Delta_t + \mu^2 \ , &(28)\\ \Delta_t &:= g^{ab}\nabla_a\nabla_b = g^{ab}\hat{X}_a\hat{X}_b - \Gamma^c_{ab}\hat{X}_c \ , &(29)\end{aligned}$$

where Δ_t is the Laplacian on Σ_t, ∇_a are the covariant derivatives corresponding to the Levi Civita connection,

$$\Gamma^\gamma_{\alpha\beta} := \frac{1}{2}\, g^{\gamma\delta}(g_{\delta\alpha,\beta} + g_{\beta\delta,\alpha} - g_{\alpha\beta,\delta} + g_{\epsilon\alpha}C^\epsilon_{\delta\beta} + g_{\epsilon\beta}C^\epsilon_{\delta\alpha}) - \frac{1}{2}\, C^\gamma_{\alpha\beta} \ , \tag{30}$$

as derived in Ref. [14], $g_{\alpha\beta,\gamma} := \hat{X}_\gamma g_{\alpha\beta}$, and $C^\gamma_{\alpha\beta}$ are the structure constants:

$$\left[\hat{X}_\alpha, \hat{X}_\beta\right] = -C^\gamma_{\alpha\beta}\,\hat{X}_\gamma\ , \tag{31}$$

with $\hat{X}_0 := \partial/\partial t$. In view of the latter equality, the structure constants with a time label vanish. This simplifies the calculations of Γ's. The only nonvanishing ones are

$$\Gamma^0_{ab} = \frac{1}{2}\,\dot{g}_{ab}\ , \tag{32}$$

$$\Gamma^c_{ab} = \frac{1}{2}\,g^{cd}(g_{ea}C^e_{db} + g_{eb}C^e_{da}) - \frac{1}{2}\,C^c_{ab}\ . \tag{33}$$

In view of these relations, the expression for $\hat{D}_1$ and $\hat{D}_2$ may be further simplified:

$$\hat{D}_1 = \frac{\partial}{\partial t}\ln\sqrt{g}\ , \tag{34}$$

$$\hat{D}_2 = -\Delta_t + \mu^2 = -(g^{ab}\hat{X}_a\hat{X}_b - C^b_{ab}g^{ac}\hat{X}_c) + \mu^2\ , \tag{35}$$

where g is the determinant of (g_{ab}). Note that for a unimodular, in particular semisimple, group $C^b_{ab} = 0$, and the second term on the right hand side of (35) vanishes. The corresponding Bianchi models are knows as Class A models.

As seen from Eq. (34), $\hat{D}_1$ acts by multiplication by a time-dependent function. Therefore, choosing $q = i\sqrt{g}$, $\hat{D}_1 = \dot{q}/q$. This reduces Eq. (11) to the eigenvalue equation

$$(\hat{D}_2 - E_n^2)\Phi_n = -(\Delta_t + E_n^2 - \mu^2)\Phi_n = 0\ , \tag{36}$$

for the operator $\hat{D}_2$ which being essentially the Laplacian over Σ_t, is self-adjoint. This guarantees the orthogonality of Φ_n and the reality of E_n^2. [3]

The analysis of the Φ_n is equivalent to the study of the eigenvectors of the Laplacian over a three-dimensional group manifold Σ_t. The general problem is the subject of the investigation in spectral geometry which is beyond the scope of the present article. However, let us recall some well-known facts about spectral properties of the Laplacian Δ for an arbitrary finite-dimensional Riemannian manifold Σ without boundary.

The following results are valid for the case where Σ is compact or the eigenfunctions are required to have a compact support[4] [15]:

1. The spectrum of Δ is an infinite discrete subset of non-negative real numbers.

[3] One can also show that, since q is imaginary, the Hamiltonian $H^{(q)}$ is self-adjoint with respect to the Klein-Gordon inner product, i.e., the inner product (11) of Ref. [1].

[4] This is equivalent with the case where Σ has a boundary $\partial\Sigma$, over which the eigenfunctions vanish.

2. The eigenvalues are either non-degenerate or finitely degenerate.

3. There is an orthonormal set of eigenfunctions which form a basis for $L^2(\Sigma)$.

4. If Σ is compact, then the first eigenvalue is zero which is non-degenerate with the eigenspace given by the set of constant functions, i.e., $\mathbb{C}$. If Σ is not compact but the eigenfunctions are required to have a compact support, then the first eigenvalue is positive.

Another piece of useful information about the spatially homogeneous cosmological models is that (up to a multiple of $i = \sqrt{-1}$) the invariant vector fields $\hat{X}_a$ yield a representation of the generators L_a of G, with $L^2(\Sigma_t)$ being the representation space, one can view the Laplacian Δ_t as (a representation of) an element of the enveloping algebra of the Lie algebra of G. Therefore, Δ_t commutes with any Casimir operator $\mathcal{C}_\lambda$ and consequently shares a set of simultaneous eigenvectors with $\mathcal{C}_\lambda$. This in turn suggests one to specialize to particular subrepresentations with definite $\mathcal{C}_\lambda$. In particular for compact groups, this leads to a reduction of the problem to a collection of finite-dimensional ones.[5]

In the remainder of this article I shall try to employ these considerations to investigate some specific models.

IV Bianchi Type I

In this case G is Abelian, therefore X_a are themselves Casimir operators and the eigenfunctions of Δ_t, i.e., Φ_n, are independent of t. Hence the Berry connection one-form (16) and its non-Abelian generalization (22) vanish identically, and the geometric phase is trivial.

V Bianchi Type IX

In this case $G = SU(2) = S^3$. The total angular momentum operator $\hat{J}^2 = \sum_a \hat{J}_a^2$ is a Casimir operator. Therefore, I shall consider the subspaces $\mathcal{H}_j$ of $\mathcal{H}_t = L^2(S_t^3)$ of definite angular momentum j. The left-invariant vector fields $\hat{X}_a$ are given by $\hat{X}_a = i\hat{J}_a$, in terms of which Eq. (31), with $C_{ab}^c = \epsilon_{abc}$, is written in the familiar form:

$$\left[\hat{J}_a, \hat{J}_b\right] = i\epsilon_{abc}\hat{J}_c \ , \tag{37}$$

with ϵ_{abc} denoting the totally antisymmetric Levi Civita symbol and $\epsilon_{123} = 1$.

[5]Here I mean a finite-dimensional Hilbert space.

Eq. (36) takes the form:

$$(\hat{H}' + k_n^2)\Phi_n = 0 \,, \tag{38}$$

where $\hat{H}'$ is an induced Hamiltonian defined by

$$\hat{H}' := g^{ab}(t)\hat{J}_a\hat{J}_b \,, \tag{39}$$

and $k_n^2 := E_n^2 - \mu^2$. Therefore, the problem of the computation of the geometric phase is identical with that of the non-relativistic quantum mechanical system whose Hamiltonian is given by (39). In particular, for the mixmaster spacetime, i.e., for g_{ab} diagonal, the problem is identical with the quantum mechanical problem of a non-relativistic asymmetric rotor, [16].

Another well-known non-relativistic quantum mechanical effect which is described by a Hamiltonian of the form (39) is the quadratic interaction of a spin with a variable electric field (E_a). The interaction potential is the Stark Hamiltonian: $\hat{H}_S = \epsilon(\sum_a E_a\hat{J}_a)^2$. The phenomenon of the geometric phase for the Stark Hamiltonian for spin $j = 3/2$, which involves Kramers degeneracy [17], was first considered by Mead [18]. Subsequently, Avron, et al [19, 20] conducted a thorough investigation of the traceless quadrupole Hamiltonians of the form (39).

The condition on the trace of the Hamiltonian is physically irrelevant, since the addition of any multiple of the identity operator to the Hamiltonian does not have any physical consequences. In general, one can express the Hamiltonian (39) in the form $\hat{H}' = \hat{\hat{H}}' + \hat{H}'_0$, where $\hat{\hat{H}}' := Tr(g^{ab})\hat{J}^2/3$,

$$\hat{H}'_0[R] := \sum_{A=1}^{5} R^A\,\hat{e}_A \,, \tag{40}$$

is the traceless part of the Hamiltonian, and

$$\begin{aligned}
\hat{e}_1 &:= J_3^2 - \frac{1}{3}\,\hat{J}^2 \,, \qquad \hat{e}_2 := \frac{1}{\sqrt{3}}\,\{\hat{J}_1,\hat{J}_2\} \,, \\
\hat{e}_3 &:= \frac{1}{\sqrt{3}}\,\{\hat{J}_2,\hat{J}_3\} \,, \qquad \hat{e}_4 := \frac{1}{\sqrt{3}}\,(\hat{J}_1^2 - \hat{J}_2^2) \,, \\
\hat{e}_5 &:= \frac{1}{\sqrt{3}}\,\{\hat{J}_1,\hat{J}_2\} \,,
\end{aligned} \tag{41}$$

$$\begin{aligned}
R^1 &:= g^{33} - \frac{1}{2}\,(g^{11}+g^{22}) \,, \qquad R^2 := \sqrt{3}\,g^{13} \,, \\
R^3 &:= \sqrt{3}\,g^{23} \,, \qquad R^4 := \frac{\sqrt{3}}{2}\,(g^{11}-g^{22}) \,, \\
R^5 &:= \sqrt{3}\,g^{12} \,.
\end{aligned} \tag{42}$$

As shown in Refs. [19, 20] the space $\mathcal{M}'$ of all traceless Hamiltonians of the form (40) is a five-dimensional real vector space. The traceless operators $\hat{e}_A$ form an orthonormal [6] basis for

[6] Orthonormality is defined by the inner product $\langle\hat{A},\hat{B}\rangle := 3\,Tr(\hat{A}\hat{B})/2$.

$\mathcal{M}'$. Removing the point ($R^A = 0$) from $\mathcal{M}'$, to avoid the collapse of all eigenvalues, one has the space $\mathbb{R}^5 - \{0\}$ as the parameter space. The situation is analogous to Berry's original example of a magnetic dipole in a changing magnetic field, [10]. Again, a rescaling of the Hamiltonian by a non-zero function of R^A does not change the geometric phase. Thus the relevant parameter space is $\mathcal{M} = S^4$. Incidentally, the point corresponding to $0 \in \mathbb{R}^5$ which is to be excluded, corresponds to the class of Friedmann-Robertson-Walker models.

Following Berry [10] and Ref. [1], let us identify t with the affine parameter of a curve $C : [0,\tau] \to S^4$ traced by the parameters R^A in S^4. Such a curve may be defined by the action of the group $SO(5)$ which acts transitively on S^4. Therefore the time-dependence of the Hamiltonian may be realized by an action of the group $SO(5)$ on a fixed Hamiltonian. As it is discussed in Ref. [20], it is the unitary representations $\mathcal{U}$ of the double cover $Spin(5) = Sp(2)$ of $SO(5)$ (alternatively the projective representations of $SO(5)$) which define the time-dependent Hamiltonian:

$$\hat{H}'_0[R(t)] = \mathcal{U}[g(t)]\, \hat{H}'_0[R(0)]\, \mathcal{U}[g(t)]^\dagger \ . \tag{43}$$

Here, $g : [0,\tau] \to Sp(2)$, is defined by $R(t) =: \pi[g(t)]R(0)$, where $\pi : Sp(2) \to SO(5)$ is the canonical two-to-one covering projection and $R(t)$ corresponds to the point $C(t) \in S^4$. The emergence of the group $Sp(2)$ is an indication of the existence of a quaternionic description of the system, [20].

Let us next examine the situation for irreducible representations j of $SU(2)$. As I previously described, $\hat{J}^2$ commutes with the Hamiltonian. Hence the Hamiltonian is block-diagonal in the basis with definite total angular momentum j. For each j, the representation space $\mathcal{H}_j$ is $2j+1$ dimensional. Therefore the restriction of the Hamiltonian $\hat{H'}_0$ to $\mathcal{H}_j$ and $\mathcal{U}[g(t)]$ are $(2j+1) \times (2j+1)$ matrices.

Let $\{\phi^I_{jn}\}$ be a complete set of orthonormal eigenvectors of the initial Hamiltonian $\hat{H}'_0[R(0)]$, where I is a degeneracy label. Then the eigenvectors of $\hat{H}'_0[R]$ are of the form

$$\Phi^I_{jn}[R] = \mathcal{U}[g]\, \phi^I_{jn} \ , \tag{44}$$

and the non-Abelian connection one-form (22) is given by

$$\mathcal{A}^{IJ}_{jn} = i\langle \phi^I_{jn}|\, \mathcal{U}^\dagger d\mathcal{U}\, |\phi^J_{jn}\rangle \ . \tag{45}$$

For the integer j, bosonic systems, it is known that the quadratic Hamiltonians of the form (40), describe time-reversal-invariant systems. In this case it can be shown that the curvature

two-form associated with the Abelian Berry connection one-form (16) vanishes identically [20]. The connection one-form is exact (gauge potential is pure gauge) and a nontrivial geometric phase can only be topological, namely it may still exist provided that the first homology group of the parameter space is nontrivial. For the problem under investigation $\mathcal{M} = S^4$, and the first homology group is trivial. Hence, in general, the Abelian geometric phase is trivial. The same conclusion cannot however be reached for the non-Abelian geometric phases.

In the remainder of this section, I shall examine the situation for some small values of j.

1) $j = 0$: The corresponding Hilbert subspace is one-dimensional. Geometric phases do not arise.

2) $j = 1/2$: In this case, $\hat{J}_a = \sigma_a/2$, where σ_a are Pauli matrices. Using the well-known anticommutation (Clifford algebra) relations $\{\sigma_a, \sigma_b\} = 2\delta_{ab}$, one can easily show that in this case

$$\hat{H}' = \frac{1}{2} \sum_a g^{aa}(t)\, \hat{I} \,, \tag{46}$$

where $\hat{I}$ is the identity matrix. Therefore, the eigenvectors Φ_n are constant (g_{ab}-independent), the connection one-form (45) vanishes, and no nontrivial geometric phases occur.

3) $j = 1$: In this case the Hilbert subspace is three-dimensional. The Abelian geometrical phases are trivial. The nontrivial matrix-valued geometrical phases may be present, provided that the Hamiltonian has a degenerate eigenvalue. Using the ordinary $j = 1$ matrix representations of $\hat{J}_a$, one can easily express $\hat{H}'_0$ as a 3×3 matrix. It can then be checked that in the generic case the eigenvalues of $\hat{H}'_0$ are not degenerate.[7] However, there are cases for which a degenerate eigenvalue is present. A simple example is the Taub metric, $(g_{ab}) = \mathrm{diag}(g_{11}, g_{22}, g_{22})$, which is a particular example of the mixmaster metric, [14]. For the general mixmaster metric, the eigenvalue problem can be easily solved. The eigenvalues of the total Hamiltonian $\hat{H}'$ of Eq. (39) are

$$\frac{1}{g_{11}} + \frac{1}{g_{22}} \,, \qquad \frac{1}{g_{11}} + \frac{1}{g_{33}} \,, \qquad \frac{1}{g_{22}} + \frac{1}{g_{33}} \,.$$

Therefore the degenerate case corresponds to the coincidence of at least two of g_{aa}'s, i.e., a Taub metric. However, even in the general mixmaster case, one can find a constant

[7] This is true for all integer j, [20].

(g_{aa}-independent) basis which diagonalizes the Hamiltonian. Hence the non-Abelian connection one-form vanishes and the geometric phase is again trivial. This is not however the case for general metrics. In the Appendix, it is shown that without actually solving the general eigenvalue problem for the general Hamiltonian, one can find the conditions on the metric which render one of the eigenvalues of the Hamiltonian degenerate. Here, I summarize the results. Using the well-known matrix representations of the angular momentum operators $\hat{J}_a$ in the $j=1$ representation [21] one can write the Hamiltonian (39) in the form:

$$\hat{H}' = \left(\begin{array}{ccc} t+2z & \xi^* & \zeta^* \\ \xi & 2(t+z) & -\xi^* \\ \zeta & -\xi & t+2z \end{array} \right) , \tag{47}$$

where

$$\begin{aligned} t &:= \frac{1}{2}\,(g^{11}+g^{22}) - g^{33} \,, \qquad z := g^{33} \,, \\ \xi &:= \frac{1}{\sqrt{2}}\,(g^{13}+ig^{23}) \,, \qquad \zeta := \frac{1}{2}\,(g^{11}-g^{22}) + ig^{12} \,. \end{aligned}$$

Then it can be shown (Appendix) that the necessary and sufficient conditions for $\hat{H}'$ to have a degenerate eigenvalue are

I. <u>for $\zeta = 0$:</u> $\xi = 0$, in which case, $\hat{H}'$ as given by Eq. (47) is already diagonal. The degenerate and non-degenerate eigenvalues are $t+2z$ and $2(t+z)$, respectively. In terms of the components of the metric, these conditions can be written as: $g_{11} = g_{22}$ and $g_{ab} = 0$ if $a \neq b$. This is a Taub metric which as discussed above does not lead to a nontrivial geometric phase.

II. <u>for $\zeta \neq 0$:</u>

$$\zeta = \mathcal{Z}\, e^{2i\theta} \,, \qquad t = \mathcal{Z} - |\xi|^2/\mathcal{Z} \,, \tag{48}$$

where $\exp[i\theta] := \xi/|\xi|$ and $\mathcal{Z} \in \mathbb{R} - \{0\}$. In this case the degenerate and non-degenerate eigenvalues are $2(\mathcal{Z}+z) - |\xi|^2/\mathcal{Z}$ and $2(z - |\xi|^2/\mathcal{Z})$, respectively.

For the latter case, an orthonormal set of eigenvectors is given by:

$$v_1 = \frac{1}{\sqrt{2}} \left(\begin{array}{c} e^{-2i\theta} \\ 0 \\ 1 \end{array} \right) , \quad v_2 = \frac{1}{\sqrt{1+2\mathcal{X}^2}} \left(\begin{array}{c} \mathcal{X} e^{-i\theta} \\ 1 \\ -\mathcal{X} e^{i\theta} \end{array} \right) , \quad v_3 = \frac{1}{\sqrt{2(1+2\mathcal{X}^2)}} \left(\begin{array}{c} -e^{-2i\theta} \\ 2\mathcal{X} e^{-i\theta} \\ 1 \end{array} \right) , \tag{49}$$

where $\mathcal{X} := |\xi|/(2\mathcal{Z})$. In view of the general argument valid for all non-degenerate eigenvalues, the geometric phase associated with v_3 is trivial. This can be directly checked

by substituting v_3 in the formula (16) for the Berry connection one-form. This leads, after some algebra, to the surprisingly simple result $\mathcal{A}_{33} := i\langle v_3|dv_3\rangle = d\theta$. Therefore, $\mathcal{A}_{33}$ is exact as expected, and the corresponding geometric phase is trivial. Similarly one can compute the matrix elements $\mathcal{A}_{rs} := i\langle v_r|dv_s\rangle$, $r, s = 1, 2$, of the non-Abelian connection one-form (22). The result is

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} 1 & \mathcal{F} \\ \mathcal{F}^* & 0 \end{pmatrix} \omega \,, \\ \mathcal{F} &:= \frac{2\mathcal{X}\, e^{i\theta}}{\sqrt{2(1+2\mathcal{X}^2)}} = \frac{2\epsilon\xi}{\sqrt{2+\left|\frac{\xi}{\zeta}\right|^2}} = \frac{\epsilon(g^{13}+ig^{23})}{\sqrt{1+\frac{(g^{13})^2+(g^{23})^2}{(g^{11}-g^{22})^2+(2g^{12})^2}}} \,, \\ \omega &:= d\theta = \frac{g^{13}dg^{23}-g^{23}dg^{13}}{(g^{13})^2+(g^{23})^2} \,, \end{aligned} \tag{50}$$

where $\epsilon := \mathcal{Z}/|\zeta| = \pm 1$. As seen from Eq. (50), $\mathcal{A}$ is a $u(2)$-valued one-form, which vanishes if g^{23}/g^{13} is kept constant during the evolution of the universe.

It is also worth mentioning that the requirement of the existence of degeneracy is equivalent to restricting the parameters of the system to a two-dimensional subset of S^4. Thus, the corresponding spectral bundle [22, 23] is a $U(2)$ vector bundle over a two-dimensional parameter space $\tilde{\mathcal{M}}$. The manifold structure of $\tilde{\mathcal{M}}$ is determined by Eqs. (48). In terms of the parameters R^A of (42), these equations are expressed by

$$R^5 = f_1\, R^4 \,, \qquad R^1 = f_2\, R^4 + \frac{f_3}{R^4} \,, \tag{51}$$

where

$$\begin{aligned} f_1 &:= \frac{2R^2R^3}{(R^2)^2-(R^3)^2} \,, \qquad f_2 := \pm\frac{(R^2)^2+(R^3)^2}{\sqrt{3}\,[(R^2)^2-(R^3)^2]} \,, \\ f_3 &:= \mp\frac{(R^2)^2-(R^3)^2}{2\sqrt{3}} \,, \qquad f_4 := (R^2)^2+(R^3)^2 \,. \end{aligned}$$

Here f_4 is also introduced for future use. In addition to (51), one also has the condition $(R^A) \in S^4$. If S^4 is identified with the round sphere, this condition takes the form $\sum_A (R^A)^2 = 1$. Substituting (51) in this equation, one finds

$$(1+f_2^2+f_3^2)(R^4)^4 - (1-f_4-2f_2f_3)(R^4)^2 + f_3^2 = 0 \,. \tag{52}$$

Eq. (52) may be easily solved for R^4. This yields

$$R^4 = \pm\frac{3[(R^2)^2-(R^3)^2]^2}{8[(R^2)^2+(R^3)^2]^2}\left[1-\frac{2}{3}[(R^2)^2+(R^3)^2] \pm \sqrt{1-\frac{4}{3}[(R^2)^2+(R^3)^2]}\right] . \tag{53}$$

Note that the parameters R^A are related to the components of the inverse of the three-metric through Eqs. (42). Thus the parameter space $\tilde{\mathcal{M}}$ is really a submanifold of the corresponding minisuperspace. Fig. 1 shows a three-dimensional plot of R^4 as a function of R^2 and R^3, i.e., a plot of the parameter space $\tilde{\mathcal{M}}$ as embedded in $\mathbb{R}^3$. Note that $R^2 = \pm R^3$ renders f_1 and f_2 singular. The corresponding points which are depicted as the curves along which the figure becomes non-differentiable must be handled with care. The smooth part of $\tilde{\mathcal{M}}$ consists of eight connected components, each of which is diffeomorphic to an open disk (alternatively $\mathbb{R}^2$).

4) $j = 3/2$: This case has been studied in Refs. [19, 20] in detail. Therefore I suffice to note that it involves nontrivial geometric phases. Note that because of Kramer's degeneracy, one does not need to restrict the minisuperspace to obtain degenerate eigenvalues. Every solution of the Bianchi type IX model involves a non-Abelian geometric phase.

VI Conclusion

In this article I applied the method developed in Ref. [1] to investigate the existence of cyclic and noncyclic adiabatic geometric phases induced by spatially homogeneous cosmological backgrounds on a complex Klein-Gordon field. Unlike the examples presented in Ref. [1], here the freedom in the choice of the decomposition parameter q turned out to simplify the analysis.

I showed that for the Bianchi type I models Berry's connection one-form vanished identically. This was not the case for the Bianchi type IX models. Hence, for these models nontrivial non-Abelian adiabatic geometric phases could occur in general. A rather interesting observation was the relationship between the induced Hamiltonians in the Bianchi type IX models and the quadrupole Hamiltonians of the molecular and nuclear physics. I also showed that even for the integer spin representations nontrivial geometric phases could exist. This should also be of interest for the molecular physicists and chemists who have apparently investigated only the fermionic systems (half-integer spin representations.) A rather thorough investigation of the non-Abelian adiabatic geometric phase for arbitrary spin 1 systems has been conducted in [24].

As described in Ref. [1] the arbitrariness in the choice of q leads to a $GL(1, \mathbb{C})$ symmetry of the two-component formulation of the Klein-Gordon equation. In the context of general relativity where the Poincaré invariance is replaced by the diffeomorphism invariance, one can

use the time-reparameterization symmetry of the background gravitational field and the geometric phase to absorb the magnitude $|q|$ of the decomposition parameter q into the definition of the lapse function $N = (-g^{00})^{-1/2}$. In this way only a $U(1)$ subgroup of the corresponding $GL(1,\mathbb{C})$ symmetry group survives. The $GL(1,\mathbb{C})$ or $U(1)$ symmetry associated with the freedom of choice of the decomposition parameter seems to have no physical basis or consequences. It is merely a mathematical feature of the two-component formalism which can occasionally be used to simplify the calculations.

The application of the two-component formulation for the Bianchi models manifestly shows that this method can be employed even for the cases where the background spacetime is non-stationary. One must however realize that the present analysis is only valid within the framework of the relativistic adiabatic approximation [1]. Although, the (approximate) stationarity of the background metric is a sufficient condition for the validity of the adiabatic approximation, it is not necessary. This can be easily seen by noting that for example in the case of Bianchi IX model, for spin $j = 1/2$ states, one has $\dot{\Phi}_n = 0$, so $\langle \Phi_m | \dot{\Phi}_n \rangle = 0$. Therefore, although the spacetime is not stationary, the adiabatic approximation yields the exact solution of the field equation. This shows that in general for arbitrary non-stationary spacetimes, there may exist adiabatically evolving states to which the above analysis applies.

The extension of our results to the non-adiabatic cases requires a generalization of the analysis of Ref. [1] to non-adiabatic evolutions.

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Appendix

In this Appendix I show how one can obtain the conditions under which the Hamiltonian (47) has degenerate eigenvalues without actually solving the eigenvalue problem in the general case.

The analysis can be slightly simplified if one writes the Hamiltonian (47) in the form:

$$\hat{H}' = (t + 2z)\hat{I} + \hat{\tilde{H}} \; ,$$

$$\hat{\tilde{H}} \;:=\; \left(\begin{array}{ccc} 0 & \xi^* & \zeta^* \\ \xi & t & -\xi^* \\ \zeta & -\xi & 0 \end{array}\right) \,, \tag{54}$$

where $\hat{I}$ is the 3×3 identity matrix. Clearly, the eigenvalue problems for $\hat{H}'$ and $\hat{\tilde{H}}$ are equivalent. Computing the characteristic polynomial for $\hat{\tilde{H}}$, i.e., $P(\lambda) := \det(\hat{\tilde{H}} - \lambda\hat{I})$, one finds:

$$P(\lambda) = -\lambda^3 + t\lambda^2 + (|\zeta|^2 + 2|\xi|^2)\lambda - (t|\zeta|^2 + \zeta\xi^{*2} + \zeta^*\xi^2)\,. \tag{55}$$

If one of the eigenvalues (roots of $P(\lambda)$) is degenerate, then

$$P(\lambda) = -(\lambda - l_1)(\lambda - l_2)^2\,. \tag{56}$$

Comparing Eqs. (55) and (56), one finds

$$t = l_1 + 2l_2\,,\quad l_2^2 + 2l_1l_2 = -(|\zeta|^2 + 2|\xi|^2)\,,\quad l_1l_2^2 = -(t|\zeta|^2 + \zeta\xi^{*2} + \zeta^8\xi^2)\,. \tag{57}$$

Furthermore since l_2 is at least doubly degenerate, the rows of the matrix:

$$\hat{\tilde{H}} - l_2\,\hat{I} = \left(\begin{array}{ccc} -l_2 & \xi^* & \zeta^* \\ \xi & t - l_2 & -\xi^* \\ \zeta & -\xi & -l_2 \end{array}\right) \,, \tag{58}$$

must be mutually linearly dependent. In other words the cofactors of all the matrix elements must vanish. Enforcing this condition for the matrix elements and using Eqs. (57), one finally finds that either $\xi = \zeta = l_2 = 0$ and $l_1 = t$, or

$$\begin{array}{rclcrcl} \zeta &=& \mathcal{Z}\, e^{2i\theta}\,, && t &=& \mathcal{Z} - \dfrac{|\xi|^2}{\mathcal{Z}}\,, \\ l_2 &=& \mathcal{Z}\,, && l_1 &=& -(\mathcal{Z} + \dfrac{|\xi|^2}{\mathcal{Z}})\,, \end{array}$$

where $\exp[i\theta] := \xi/|\xi|$ and $\mathcal{Z} \in \mathbb{R} - \{0\}$.

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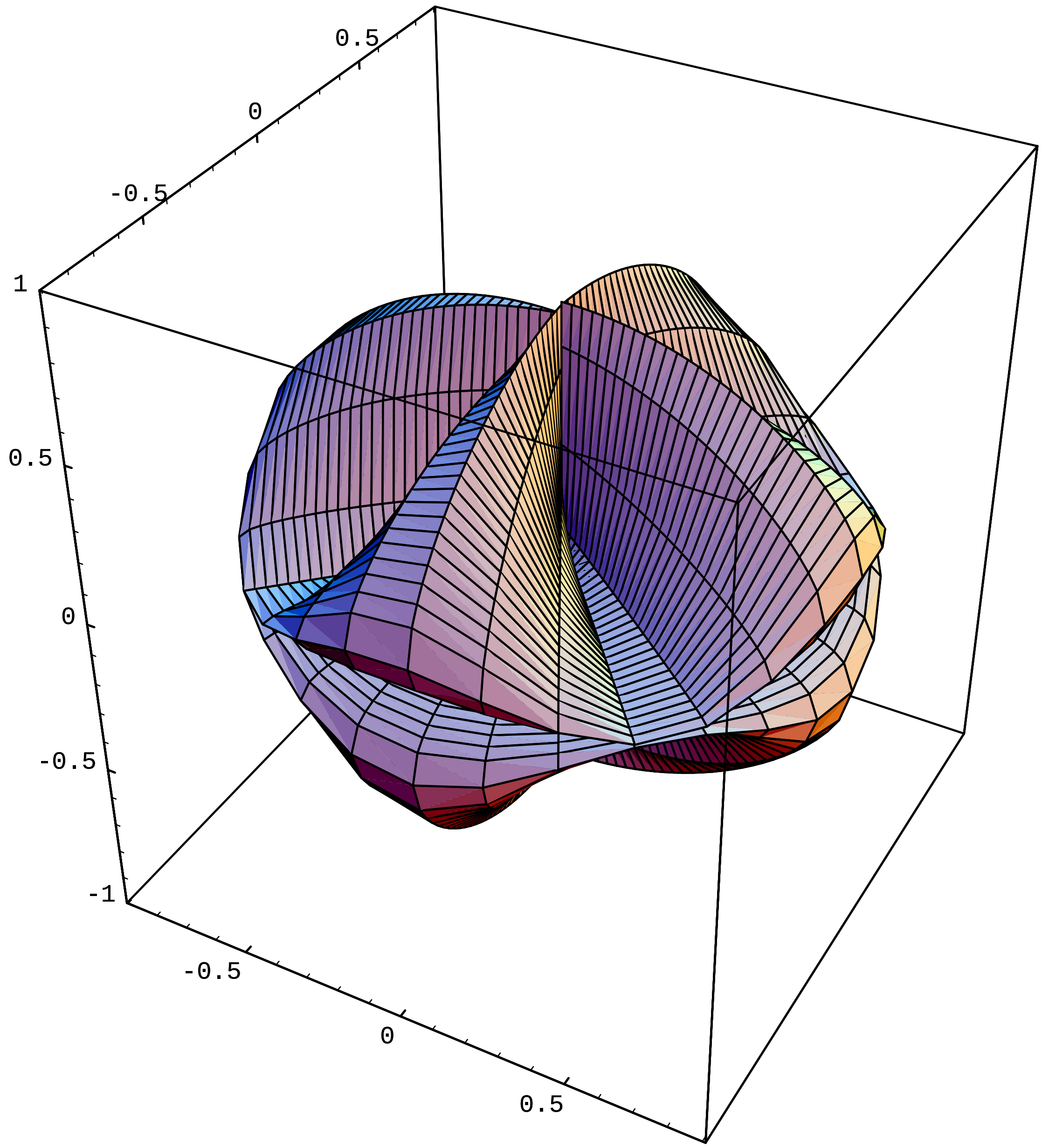
Figure Caption:

Figure 1: This is a plot of $R^4 = R^4(R^2, R^3)$. The horizontal plane is the R^2-R^3-plane and the vertical axis is the R^4-axis. The parameter space $\tilde{\mathcal{M}}$ is obtained by removing the intersection of this figure with the planes defined by: $R^4 = 0$, $R^2 = R^3$ and $R^2 = -R^3$. The intersection involves the curves along which the figure becomes non-differentiable.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